

More on Models and Numerical Procedures

Chapter 26

Three Alternatives to Geometric Brownian Motion

- Constant elasticity of variance (CEV)
- Mixed Jump diffusion
- Variance Gamma

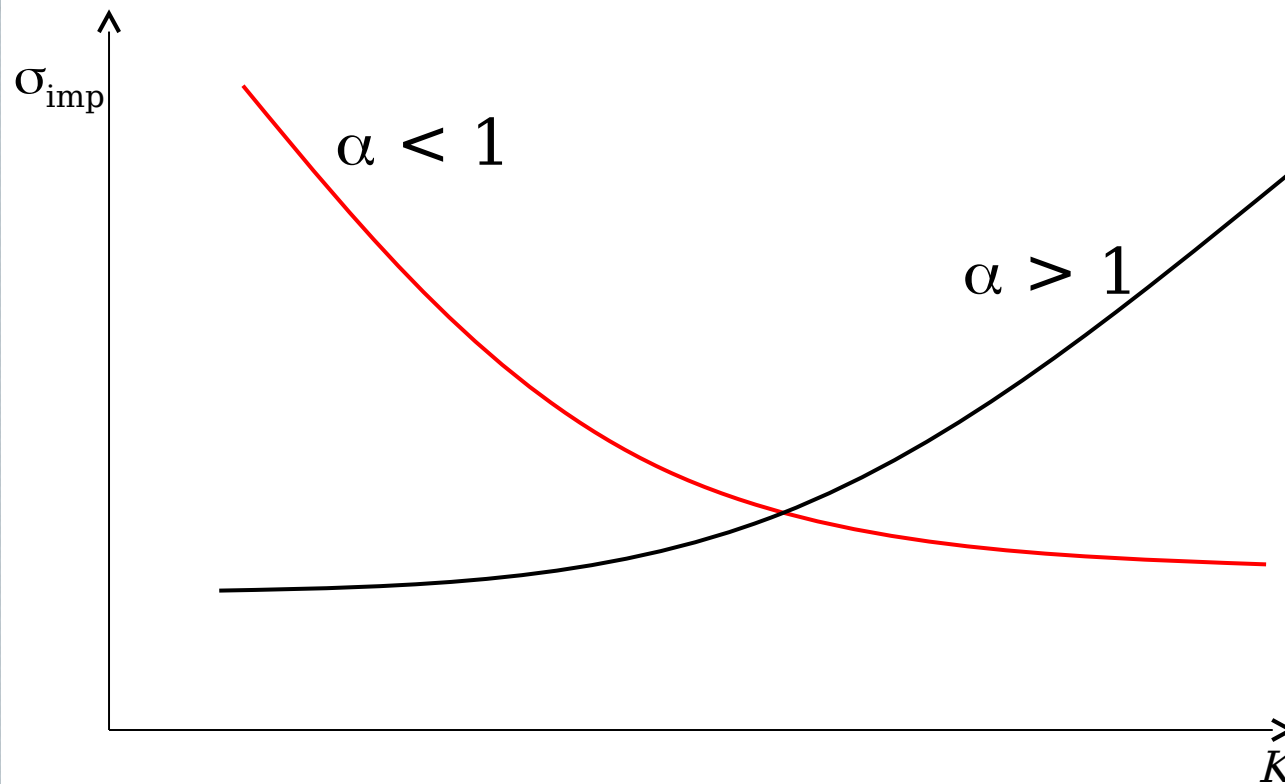
CEV Model (page 592 to 593)

$$dS = (r - q)Sdt + \sigma S^\alpha dz$$

- When $\alpha = 1$ the model is Black-Scholes
- When $\alpha > 1$ volatility rises as stock price rises
- When $\alpha < 1$ volatility falls as stock price rises

European option can be value analytically in terms of the cumulative non-central chi square distribution

CEV Models Implied Volatilities



Mixed Jump Diffusion Model (page 593 to 594)

Merton produced a pricing formula when the asset price follows a diffusion process overlaid with random jumps

- dp is the random jump
- k is the expected size of the jump
- λdt is the probability that a jump occurs in the next interval of length dt

$$dS/S = (r - q - \lambda k)dt + \sigma dz + dp$$

Jumps and the Smile

- Jumps have a big effect on the implied volatility of short term options
- They have a much smaller effect on the implied volatility of long term options

The Variance-Gamma Model

(page 594 to 595)

- Define g as change over time T in a variable that follows a gamma process. This is a process where small jumps occur frequently and there are occasional large jumps
- Conditional on g , $\ln S_T$ is normal. Its variance proportional to g
- There are 3 parameters
 - v , the variance rate of the gamma process
 - σ^2 , the average variance rate of $\ln S$ per unit time
 - θ , a parameter defining skewness

Understanding the Variance-Gamma Model

- g defines the rate at which information arrives during time T (g is sometimes referred to as measuring economic time)
- If g is large the change in $\ln S$ has a relatively large mean and variance
- If g is small relatively little information arrives and the change in $\ln S$ has a relatively small mean and variance

Time Varying Volatility

- Suppose the volatility is σ_1 for the first year and σ_2 for the second and third
- Total accumulated variance at the end of three years is $\sigma_1^2 + 2\sigma_2^2$
- The 3-year average volatility is

$$3\bar{\sigma}^2 = \sigma_1^2 + 2\sigma_2^2; \quad \bar{\sigma} = \sqrt{\frac{\sigma_1^2 + 2\sigma_2^2}{3}}$$

Stochastic Volatility

Models (equations 26.2 and 26.3, page 597)

$$\frac{dS}{S} = (r - q)dt + \sqrt{V}dz_S$$

$$dV = a(V_L - V)dt + \xi V^\alpha dz_V$$

When V and S are uncorrelated a European option price is the Black-Scholes price integrated over the distribution of the average variance

Stochastic Volatility Models

continued

- When V and S are negatively correlated we obtain a downward sloping volatility skew similar to that observed in the market for equities
- When V and S are positively correlated the skew is upward sloping. (This pattern is sometimes observed for commodities)

The IVF Model (page 598)

The implied volatility function model is designed to create a process for the asset price that exactly matches observed option prices. The usual geometric Brownian motion model

$$dS = (r - q)Sdt + \sigma Sdz$$

is replaced by

$$dS = [r(t) - q(t)]Sdt + \sigma(S, t)Sdz$$

The Volatility Function

(equation 26.4)

The volatility function that leads to the model matching all European option prices is

$$[\sigma(K, t)]^2 = \frac{2 \frac{\partial c_{mkt}}{\partial t} + q(t) c_{mkt} + K[r(t) - q(t)] \frac{\partial c_{mkt}}{\partial K}}{K^2 (\partial^2 c_{mkt} / \partial K^2)}$$

Strengths and Weaknesses of the IVF Model

- The model matches the probability distribution of asset prices assumed by the market at each future time
- The model does not necessarily get the joint probability distribution of asset prices at two or more times correct

Convertible Bonds

- Often valued with a tree where during a time interval Δt there is
 - a probability p_u of an up movement
 - A probability p_d of a down movement
 - A probability $1 - \exp(-\lambda t)$ that there will be a default (λ is the hazard rate)
- In the event of a default the stock price falls to zero and there is a recovery on the bond

The Probabilities

$$p_u = \frac{a - d\bar{e}^{\lambda\Delta t}}{u - d}$$

$$p_d = \frac{u\bar{e}^{\lambda\Delta t} - a}{u - d}$$

$$u = e^{\sqrt{(\sigma^2 - \lambda)\Delta t}}$$

$$d = \frac{1}{u}$$

Node Calculations

Define:

Q_1 : value of bond if neither converted nor called

Q_2 : value of bond if called

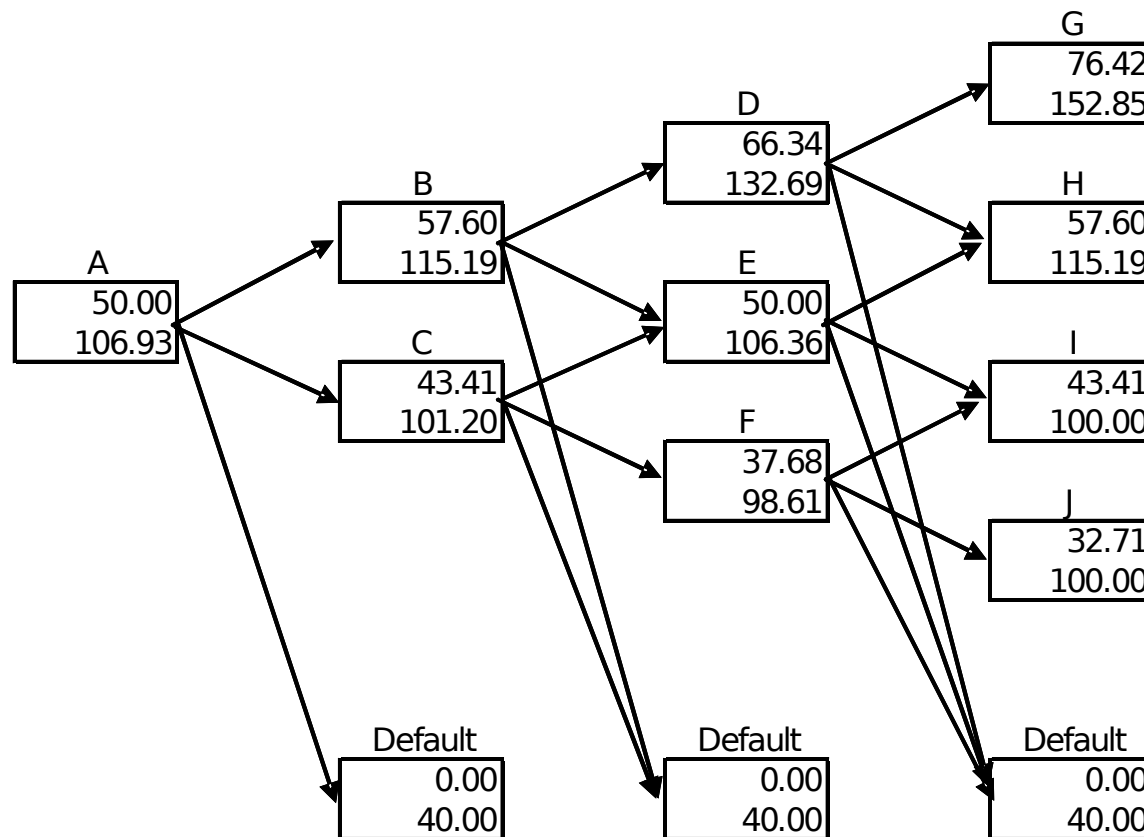
Q_3 : value of bond if converted

Value at a node = $\max[\min(Q_1, Q_2), Q_3]$

Example 26.1 (page 600)

- 9-month zero-coupon bond with face value of \$100
- Convertible into 2 shares
- Callable for \$113 at any time
- Initial stock price = \$50,
- volatility = 30%,
- no dividends
- Risk-free rates all 5%
- Default intensity, λ , is 1%
- Recovery rate=40%

The Tree (Figure 26.2, page 601)



Numerical Procedures

Topics:

- Path dependent options using tree
- Barrier options
- Options where there are two stochastic variables
- American options using Monte Carlo

Path Dependence: The Traditional View

- Backwards induction works well for American options. It cannot be used for path-dependent options
- Monte Carlo simulation works well for path-dependent options; it cannot be used for American options

Extension of Backwards Induction

- Backwards induction can be used for some path-dependent options
- We will first illustrate the methodology using lookback options and then show how it can be used for Asian options

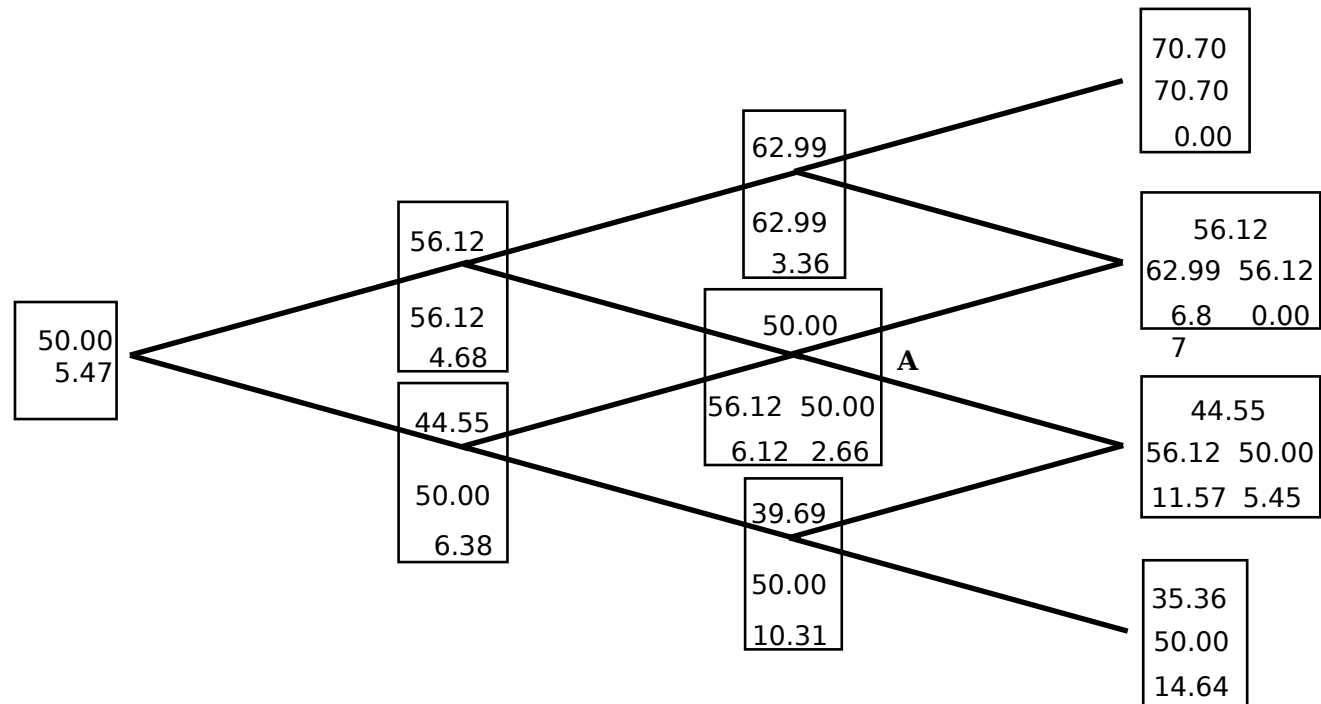
Lookback Example (Page 602-603)

- Consider an American lookback put on a stock where
 $S = 50$, $\sigma = 40\%$, $r = 10\%$, $\Delta t = 1$ month & the life of the option is 3 months
- Payoff is $S_{\max} - S_T$
- We can value the deal by considering all possible values of the maximum stock price at each node

(This example is presented to illustrate the methodology. It is not the most efficient way of handling American lookbacks (See Technical Note 13))

Example: An American Lookback Put Option (Figure 26.3, page 603)

$S_0 = 50$, $\sigma = 40\%$, $r = 10\%$, $\Delta t = 1$ month,



Why the Approach Works

This approach works for lookback options because

- The payoff depends on just 1 function of the path followed by the stock price. (We will refer to this as a “path function”)
- The value of the path function at a node can be calculated from the stock price at the node & from the value of the function at the immediately preceding node
- The number of different values of the path function at a node does not grow too fast as we increase the number of time steps on the tree

Extensions of the Approach

- The approach can be extended so that there are no limits on the number of alternative values of the path function at a node
- The basic idea is that it is not necessary to consider every possible value of the path function
- It is sufficient to consider a relatively small number of representative values of the function at each node

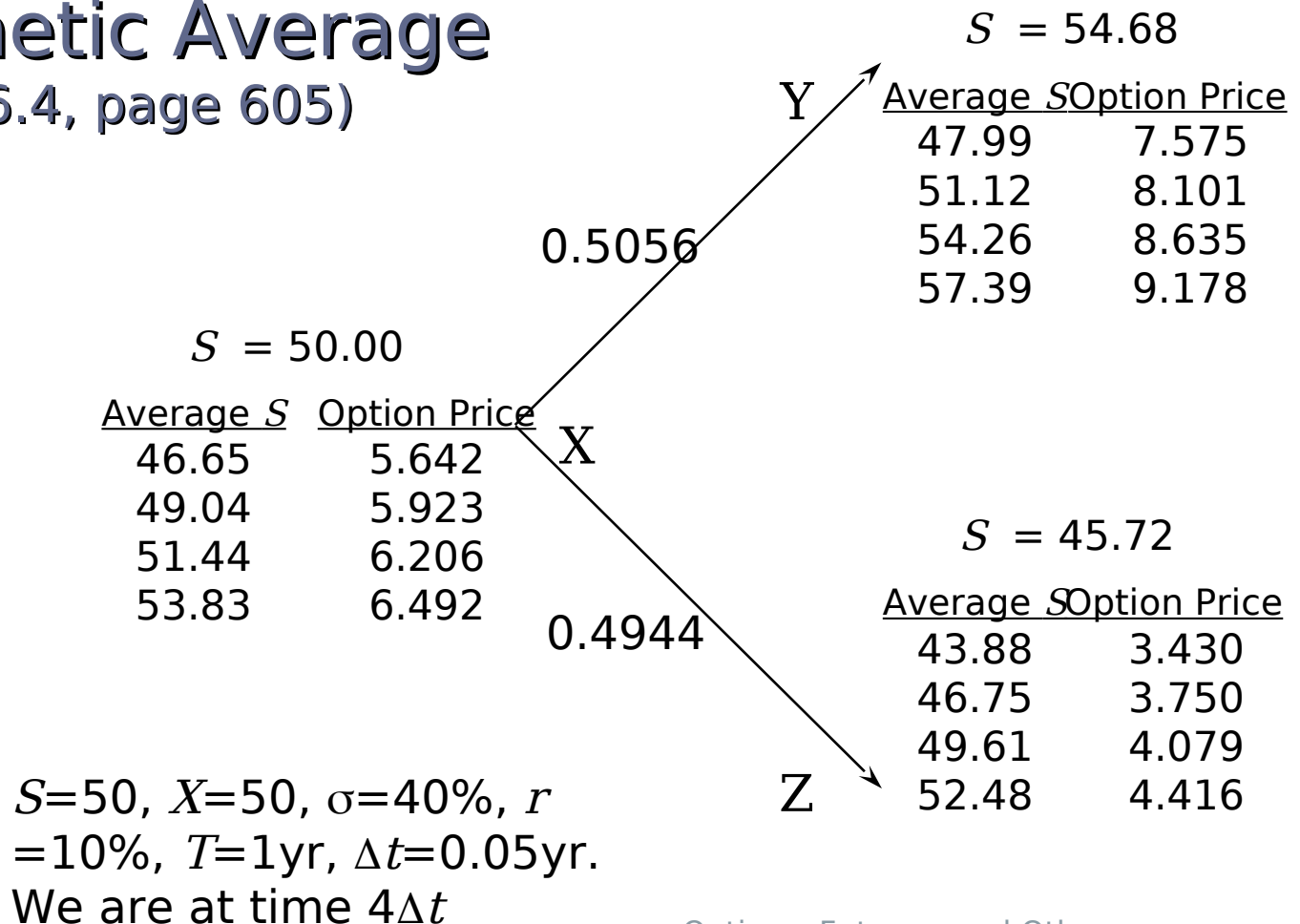
Working Forward

- First work forward through the tree calculating the max and min values of the “path function” at each node
- Next choose representative values of the path function that span the range between the min and the max
 - Simplest approach: choose the min, the max, and N equally spaced values between the min and max

Backwards Induction

- We work backwards through the tree in the usual way carrying out calculations for each of the alternative values of the path function that are considered at a node
- When we require the value of the derivative at a node for a value of the path function that is not explicitly considered at that node, we use linear or quadratic interpolation

Part of Tree to Calculate Value of an Option on the Arithmetic Average (Figure 26.4, page 605)



Part of Tree to Calculate Value of an Option on the Arithmetic Average (continued)

Consider Node X when the average of 5 observations is 51.44

Node Y: If this is reached, the average becomes 51.98. The option price is interpolated as 8.247

Node Z: If this is reached, the average becomes 50.49. The option price is interpolated as 4.182

Node X: value is

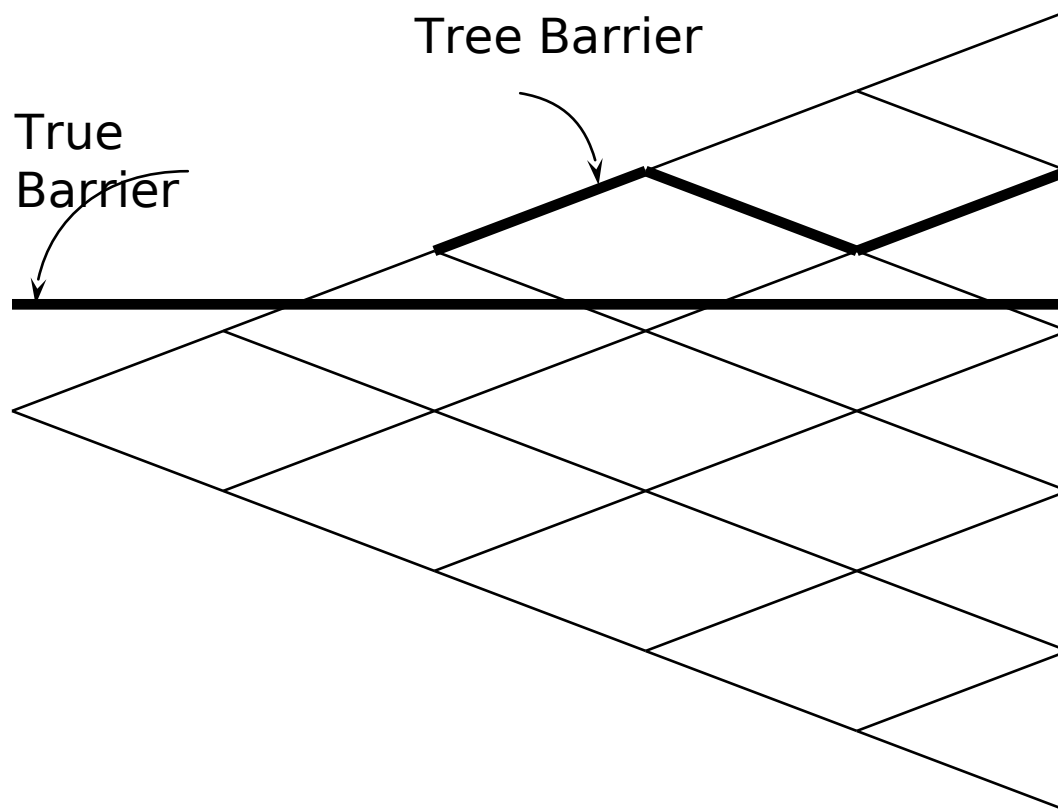
$$(0.5056 \times 8.247 + 0.4944 \times 4.182)e^{-0.1 \times 0.05} = 6.206$$

Using Trees with Barriers

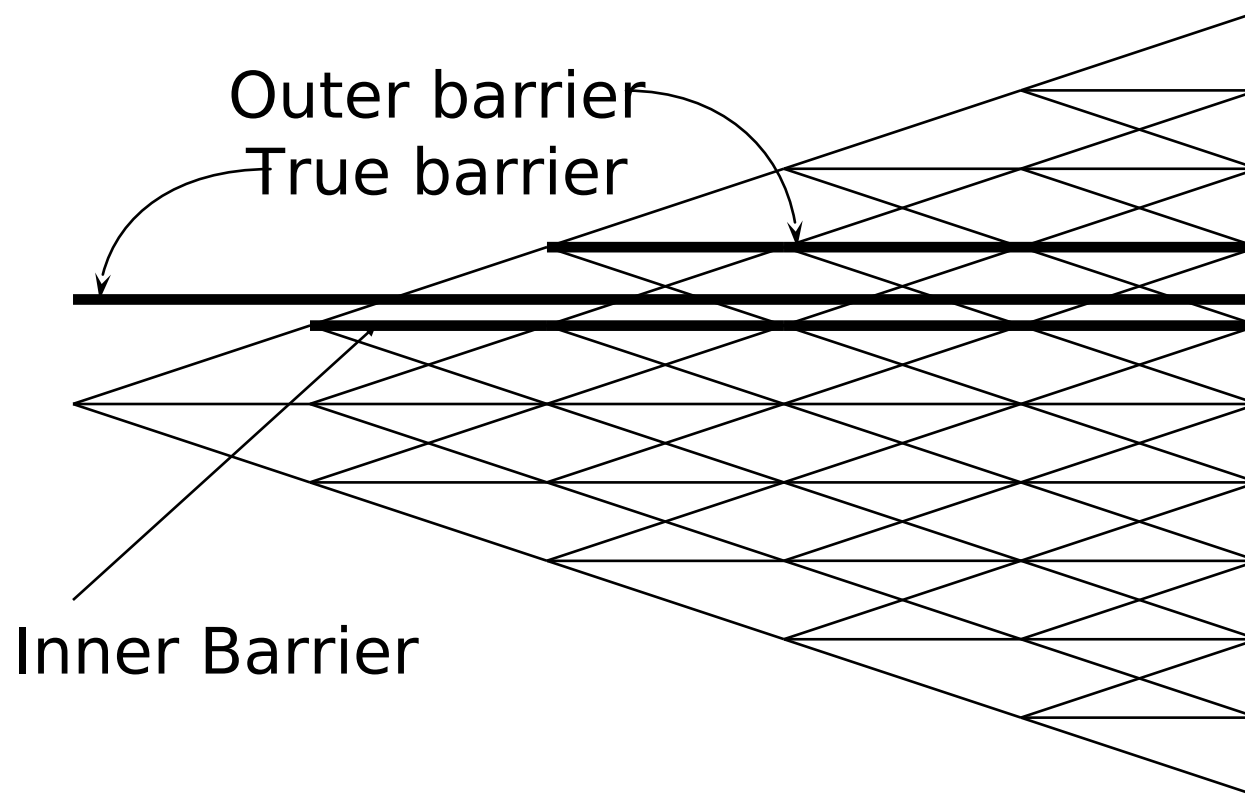
(Section 26.6, page 606)

- When trees are used to value options with barriers, convergence tends to be slow
- The slow convergence arises from the fact that the barrier is inaccurately specified by the tree

True Barrier vs Tree Barrier for a Knockout Option: The Binomial Tree Case



Inner and Outer Barriers for Trinomial Trees (Figure 26.4, page 607)



Alternative Solutions to Valuing Barrier Options

- Interpolate between value when inner barrier is assumed and value when outer barrier is assumed
- Ensure that nodes always lie on the barriers
- Use adaptive mesh methodology

In all cases a trinomial tree is preferable to a binomial tree

Modeling Two Correlated Variables

(Section 26.7, page 609)

APPROACHES:

1. Transform variables so that they are not correlated & build the tree in the transformed variables
2. Take the correlation into account by adjusting the position of the nodes
3. Take the correlation into account by adjusting the probabilities

Monte Carlo Simulation and American Options

- Two approaches:
 - The least squares approach
 - The exercise boundary parameterization approach
- Consider a 3-year put option where the initial asset price is 1.00, the strike price is 1.10, the risk-free rate is 6%, and there is no income

Sampled Paths

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

The Least Squares Approach (page 612)

- We work back from the end using a least squares approach to calculate the continuation value at each time
- Consider year 2. The option is in the money for five paths. These give observations on S of 1.08, 1.07, 0.97, 0.77, and 0.84. The continuation values are 0.00 , $0.07e^{-0.06}$, $0.18e^{-0.06}$, $0.20e^{-0.06}$, and $0.09e^{-0.06}$

The Least Squares Approach continued

- Fitting a model of the form $V=a+bS+cS^2$ we get a best fit relation
$$V=-1.070+2.983S-1.813S^2$$
for the continuation value V
- This defines the early exercise decision at $t=2$. We carry out a similar analysis at $t=1$

The Least Squares

Approach continued

In practice more complex functional forms can be used for the continuation value and many more paths are sampled

The Early Exercise Boundary Parametrization Approach (page 615)

- We assume that the early exercise boundary can be parameterized in some way
- We carry out a first Monte Carlo simulation and work back from the end calculating the optimal parameter values
- We then discard the paths from the first Monte Carlo simulation and carry out a new Monte Carlo simulation using the early exercise boundary defined by the parameter values.

Application to Example

- We parameterize the early exercise boundary by specifying a critical asset price, S^* , below which the option is exercised.
- At $t = 1$ the optimal S^* for the eight paths is 0.88. At $t = 2$ the optimal S^* is 0.84
- In practice we would use many more paths to calculate the S^*